Validity of the Gell-Mann formula for $sl(n,\mathbb{R})$ and $su(n)$ algebras *

Igor Salom †
Institute of Physics, P.O. Box 57, 11001 Belgrade, Serbia

Djordje Šijački ‡
Institute of Physics, P.O. Box 57, 11001 Belgrade, Serbia

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Abstract

The so called Gell-Mann formula, a prescription designed to provide an inverse to the Inönü-Wigner Lie algebra contraction, has a great versatility and potential value. This formula has no general validity as an operator expression. The question of applicability of Gell-Mann's formula to various algebras and their representations was only partially treated. The validity constraints of the Gell-Mann formula for the case of $sl(n,\mathbb{R})$ and $su(n)$ algebras are clarified, and the complete list of representations spaces for which this formula applies is given. Explicit expressions of the $sl(n,\mathbb{R})$ generators matrix elements are obtained, in these cases, by making use of the Gell-Mann formula.

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1 Introduction

The Gell-Mann formula [1, 2, 3, 4] is a prescription aimed to serve as an "inverse" to the Inönü-Wigner contraction [5]. Let a symmetric Lie algebra

* This work was supported in part by MNTR, Belgrade, Project-141036.
† e-mail address: isalom@phy.bg.ac.rs
‡ e-mail address: sijacki@phy.bg.ac.rs

$\mathcal{A} = \mathcal{M} + \mathcal{T}$:

$$[\mathcal{M}, \mathcal{M}] \subset \mathcal{M}, \quad [\mathcal{M}, \mathcal{T}] \subset \mathcal{T}, \quad [\mathcal{T}, \mathcal{T}] \subset \mathcal{M}, \tag{1}$$

and its Inönü-Wigner contraction $\mathcal{A}' = \mathcal{M} + \mathcal{U}$:

$$[\mathcal{M}, \mathcal{M}] \subset \mathcal{M}, \quad [\mathcal{M}, \mathcal{U}] \subset \mathcal{U}, \quad [\mathcal{U}, \mathcal{U}] = \{0\}, \tag{2}$$

be given. The Gell-Mann formula prescribes that elements $T \in \mathcal{T}$ can be, loosely speaking, constructed as the following simple function of the contracted algebra operators $U \in \mathcal{U}$ and $M \in \mathcal{M}$:

$$T = i\frac{\alpha}{\sqrt{U \cdot U}}[C_2(\mathcal{M}), U] + \sigma U. \tag{3}$$

Here, $C_2(\mathcal{M})$ denotes the second order Casimir operator of the $\mathcal{M}$ subalgebra, α is a normalization constant and σ is an arbitrary parameter. This formula was introduced by Dothan and Ne'eman [6] and advocated by Hermann. For a mathematically strict definition, cf. [1].

This formula is of a great potential value due to its simplicity and the fact that many aspects of the representation theory are much simpler for the contracted groups/algebras (e.g. construction of representations [7], decompositions of a direct product of representations [2], etc.). However, this formula is valid, on the algebraic level, only in the case of contraction from $\mathcal{A} = so(m+1, n)$ and/or $\mathcal{A} = so(m, n+1)$ to $\mathcal{A}' = iso(m, n)$, with $\mathcal{M} = so(m, n)$ [8, 9]. Moreover, apart from this, the formula is also partially applicable in a broad class of other contractions provided one restricts to some classes of the algebra representations. The validity of Gell-Mann's formula in a weak sense, when an algebra representation requirement is imposed as well, was investigated long ago by Hermann [2, 3]. A partial set of classes of the algebra representations for which the Gell-Mann formula holds is listed [3]. No attempt to make this list exhaustive is made, deliberately concentrating "on what seems to be the simplest situation". This analysis excluded, from the very beginning, the cases of representations where the little group (in Wigner's terminology) is non-trivially represented, not claiming a complete answer even than.

The aim of this paper is to clarify the matters of the Gell-Mann formula applicability for the class of $sl(n, \mathbb{R})$ algebras contracted w.r.t. their $so(n)$ maximal compact subalgebras. Note, that owing to a direct connection of the $sl(n, \mathbb{R})$ and $su(n)$ algebras, the conclusions readily convey to the latter

case. Apart from pure group-theoretical reasons, this problem is strongly motivated by physical applications of the Gell-Mann formula to the areas of gravity and pD-brane physics. The representations of the $SL(n,\mathbb{R})$ groups and their algebras, and, in particular, of their double coverings $\overline{SL}(n,\mathbb{R})$ (whose spinorial representations are necessarily infinite-dimensional), are of interest [10, 11]. In these applications, the solution of the labeling problem of the $\overline{SL}(n,\mathbb{R})$ groups representations is only a starting point. By a rule, a detailed information about the matrix elements of the noncompact operators in a basis of the maximal compact subgroup $Spin(n)$, a representation content of the $Spin(n)$ sub-representations etc. is required. The Gell-Mann formula offers a powerful method to describe various representation details in a simple closed analytic form.

2 Framework

In this paper, rather then following the approach of Hermann [3], we work in the representation space of square integrable functions over the maximal compact subgroup $Spin(n)$, with a standard invariant Haar measure: $\mathcal{L}^2(Spin(n))$. This representation space is large enough to provide for all inequivalent irreducible representations of the contracted group, and is also rich enough to contain representatives from all equivalence classes of the $\overline{SL}(n,\mathbb{R})$ group, i.e. $sl(n,\mathbb{R})$ algebra, representations [12].

The generators of the contracted group are generically represented, in this space, as follows. The $so(n)$ subalgebra operators act, in a standard way, via a right group action:

$$M_{ab}\left|\phi\right\rangle = -i\frac{d}{dt}\exp(itM_{ab})\Big|_{t=0}\left|\phi\right\rangle, \quad g'\left|g\right\rangle = \left|g'g\right\rangle, \quad \left|\phi\right\rangle \in \mathcal{L}^2(Spin(n)).$$

The Abelian operators U_μ act multiplicatively as Wigner D-functions (the $SO(n)$ group matrix elements expressed as functions of the group parameters):

$$U_\mu \to |u| D^{\square\!\square}_{x\mu}(g^{-1}) \equiv \left\langle \begin{matrix}\square\!\square \\ x\end{matrix} \,\middle|\, g^{-1} \,\middle|\, \begin{matrix}\square\!\square \\ \mu\end{matrix} \right\rangle, \tag{4}$$

$|u|$ being a constant norm, g being an $SO(n)$ element, and $\square\!\square$ denoting the symmetric second order tensor representation of $SO(n)$. The $\left|\begin{matrix}\square\!\square \\ \mu\end{matrix}\right\rangle$ vector from representation $\square\!\square$ space is denoted by the index of the operator

U_μ, whereas the vector $\left| \begin{array}{c} \square\square \\ x \end{array} \right\rangle$ can be an arbitrary vector belonging to $\square\square$ (the choice of x determines, in Wigner terminology, the little group of the representation in question). Taking an inverse of g in (4) insures the correct transformation properties.

A natural discrete orthonormal basis in the $Spin(n)$ representation space is given by properly normalized functions of the $Spin(n)$ representation matrix elements:

$$\left\{ \left| \begin{array}{c} J \\ km \end{array} \right\rangle \equiv \int \sqrt{dim(J)} D^J_{km}(g^{-1}) dg \, |g\rangle \right\}, \ \left\langle \begin{array}{c} J \\ km \end{array} \right| \left. \begin{array}{c} J' \\ k'm' \end{array} \right\rangle = \delta_{JJ'} \delta_{kk'} \delta_{mm'}, \tag{5}$$

where dg is an (normalized) invariant Haar measure. Here, J stands for a set of $Spin(n)$ irreducible representation labels, while the k and m labels numerate the representation basis vectors.

An action of the $so(n)$ operators in this basis is well known, and it can be written in terms of the Clebsch-Gordan coefficients of the $Spin(n)$ group as follows,

$$\langle M_{ab} \rangle = \left\langle \begin{array}{c} J' \\ k'm' \end{array} \right| M_{ab} \left| \begin{array}{c} J \\ km \end{array} \right\rangle = \delta_{JJ'} \sqrt{C_2(J)} \, C^{J \ \square \ J'}_{m (ab) m'} \ . \tag{6}$$

The matrix elements of the U_μ operators in this basis are readily found to read:

$$\left\langle U^{(x)}_\mu \right\rangle = |u| \left\langle \begin{array}{c} J' \\ k'm' \end{array} \right| D^{-1 \square\square}_{x\mu} \left| \begin{array}{c} J \\ km \end{array} \right\rangle = |u| \sqrt{\tfrac{dim(J)}{dim(J')}} C^{J \ \square\square \ J'}_{k \ x \ k'} \ C^{J \ \square\square \ J'}_{m (ab) m'} \ . \tag{7}$$

A closed form of the matrix elements of the whole contracted algebra $r_{\frac{n(n+1)}{2}-1} \uplus so(n)$ (a semidirect sum of a $\frac{n(n+1)}{2}-1$ dimensional Abelian algebra and $so(n)$) representations is thus explicitly given in this space by (6) and (7).

Moreover, we introduce the so called, left action generators K as:

$$K_\mu \equiv g^{\nu\lambda} D^{\square}_{\mu\nu} M_\lambda, \tag{8}$$

where $g^{\nu\lambda}$ is the Cartan metric tensor of $SO(n)$. The K_μ operators behave exactly as the rotation generators M_μ, it is only that they act on the lower left-hand side indices of the basis (5):

$$\langle K_{ab} \rangle = \left\langle \begin{array}{c} J' \\ k'm' \end{array} \right| K_{ab} \left| \begin{array}{c} J \\ km \end{array} \right\rangle = \delta_{JJ'} \sqrt{C_2(J)} \, C^{J \ \square \ J'}_{k (ab) k'} \ . \tag{9}$$

The operators K_μ and M_μ mutually commute. However, the corresponding Casimir operators match and, in particular, we will use $\sum K_\mu^2 = \sum M_\mu^2$ in the expression for the Gell-Mann formula (3).

3 Validity of the Gell-Mann formula

The Gell-Mann formula validity problem is due to the fact that the third commutation relation of (1) is not a priori satisfied as an operator relation when the algebra elements are given by expressions (3). In the $sl(n,\mathbb{R})$ case, the $\mathcal{T}$ subspace is spanned by $\frac{1}{2}n(n+1)-1$ of, so called, shear generators T_μ. These operators transform as a second order symmetric tensor w.r.t. $Spin(n)$ subgroup, and, in the Cartesian basis, satisfy:

$$[T_{ab}, T_{cd}] = i(\delta_{ac}M_{db} + \delta_{ad}M_{cb} + \delta_{bc}M_{da} + \delta_{bd}M_{ca}). \tag{10}$$

To investigate circumstances in which this relation holds, we evaluate the commutator of two shear generators in the framework given in previous section. The Gell-Mann formula (3) reads now:

$$T_\mu = i\alpha[C_2(so(n))_K, D^{\square\square}_{x\mu}] + \sigma D^{\square\square}_{x\mu}, \tag{11}$$

where $C_2(so(n)_K) = \frac{1}{2}\sum_{a,b=1}^{n}(K_{ab})^2$. By making use of this formula, a few algebraic relations and some properties of the Wigner D-functions, after some algebra we obtain:

$$\begin{aligned}
[T_\mu, T_\nu] &= -2\alpha^2\big[K_{\{i}, [K_{j\}}, D^{\square\square}_{x\nu}]\big][K_j, D^{\square\square}_{x\mu}]K_i - (\mu \leftrightarrow \nu) \\
&= \cdots = -\alpha^2 \textstyle\sum_J \sum_{\lambda,\lambda'} \big(C^{\square\square\ \square\square\ J}_{\ \mu\ \ \ \ \nu\ \ \ \lambda} - C^{\square\square\ \square\square\ J}_{\ \nu\ \ \ \ \mu\ \ \ \lambda}\big)\cdot \\
&\quad \Big(2\big(C_2(J) - 2C_2(\square\square)\big)\big\langle\big\langle{}^{J}_{\lambda'}\big|1\otimes K_i\big|{}^{\square\square}_{\ x}\big\rangle\big|{}^{\square\square}_{\ x}\big\rangle + \\
&\quad \big\langle\big\langle{}^{J}_{\lambda'}\big|[1\otimes K_i, C_{2(I+II)_K}]\big|{}^{\square\square}_{\ x}\big\rangle\big|{}^{\square\square}_{\ x}\big\rangle\Big) D^J_{\lambda'\lambda}K_i,
\end{aligned} \tag{12}$$

where a summation over repeated Latin indices i and j that label the K generators in any real basis (such that $C_{2_K} = K_iK_i$) is assumed. The $C_{2(I+II)_K}$ operator here denotes the second order Casimir operator acting in the tensor product of two $\square\square$ representations, i.e. $C_{2(I+II)_K} = \sum_i (K_i \otimes 1 + 1 \otimes K_i)^2$.

The summation index J in (12) runs over all irreducible representations of the $Spin(n)$ group that appear in the tensor product $\square\square \otimes \square\square$, and λ, λ'

count the vectors of these representations. Since all irreducible representations terms, apart those for which the Clebsch-Gordan coefficient $C^{\square\!\square\;\square\!\square\;J}_{\mu\;\;\nu\;\;\lambda}$ is antisymmetric w.r.t. $\mu \leftrightarrow \nu$ vanish, we are left with only two values that J takes: one corresponding to the antisymmetric second order tensor $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ and the other one corresponding to the representation that we denote as $\begin{smallmatrix}\square\!\square\\\square\!\square\end{smallmatrix}$. The fact that in the case of $sl(n,\mathbb{R})$ algebras, there is another representation term, in addition to $\begin{smallmatrix}\square\\\square\end{smallmatrix}$, in the antisymmetric product of two $\square\!\square$ representations (i.e. representations that correspond to abelian U operators), is in the root of the Gell-Mann formula validity problem. Note that in the case of the $so(m+1,n) \rightarrow iso(m,n)$, i.e. $so(m,n+1) \rightarrow iso(m,n)$) contractions, where the Gell-Mann formula works on the algebraic level, the contracted U operators transform as $\square$ and the antisymmetric product of two such representations certainly belongs to the $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ representation and closes upon the $\mathcal{M} = so(m,n)$ subalgebra.

The $so(n)$ Casimir operator values satisfy $C_2(\begin{smallmatrix}\square\!\square\\\square\!\square\end{smallmatrix}) = 2C_2(\square\!\square) = 4n$, implying that one of the two terms vanishes in (12) when $J = \begin{smallmatrix}\square\!\square\\\square\!\square\end{smallmatrix}$, leaving us with:

$$\begin{aligned}\tfrac{1}{2\alpha^2}[T_\mu, T_\nu] = 4(n+2)\textstyle\sum_{\lambda,\lambda'} C^{\square\!\square\;\square\!\square\;\begin{smallmatrix}\square\\\square\end{smallmatrix}}_{\mu\;\;\nu\;\;\lambda}\Big\langle\Big\langle \begin{smallmatrix}\square\\\square\end{smallmatrix}_{\lambda'}\Big| 1\otimes K_i \Big|\begin{smallmatrix}\square\!\square\end{smallmatrix}_x\Big\rangle\Big|\begin{smallmatrix}\square\!\square\end{smallmatrix}_x\Big\rangle D^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}_{\lambda'\lambda} K_i - \\ \textstyle\sum_{\lambda,\lambda'} C^{\square\!\square\;\square\!\square\;\begin{smallmatrix}\square\\\square\end{smallmatrix}}_{\mu\;\;\nu\;\;\lambda}\Big\langle\Big\langle \begin{smallmatrix}\square\\\square\end{smallmatrix}_{\lambda'}\Big|[1\otimes K_i, C_{2(I+II)_K}]\Big|\begin{smallmatrix}\square\!\square\end{smallmatrix}_x\Big\rangle\Big|\begin{smallmatrix}\square\!\square\end{smallmatrix}_x\Big\rangle D^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}_{\lambda'\lambda} K_i - \\ \textstyle\sum_{\lambda,\lambda'} C^{\square\!\square\;\square\!\square\;\begin{smallmatrix}\square\!\square\\\square\!\square\end{smallmatrix}}_{\mu\;\;\nu\;\;\lambda}\Big\langle\Big\langle \begin{smallmatrix}\square\!\square\\\square\!\square\end{smallmatrix}_{\lambda'}\Big|[1\otimes K_i, C_{2(I+II)_K}]\Big|\begin{smallmatrix}\square\!\square\end{smallmatrix}_x\Big\rangle\Big|\begin{smallmatrix}\square\!\square\end{smallmatrix}_x\Big\rangle D^{\begin{smallmatrix}\square\!\square\\\square\!\square\end{smallmatrix}}_{\lambda'\lambda} K_i,\end{aligned} \tag{13}$$

where we used that $C_2(\begin{smallmatrix}\square\\\square\end{smallmatrix}) = 2n - 4$.

As the coefficient α can be adjusted freely, all that is needed for the Gell-Mann formula to be valid is that (13) is proportional to the appropriate linear combination of the $Spin(n)$ generators, as determined by the Wigner-Eckart theorem, i.e.:

$$[T_\mu, T_\nu] \sim \sum_\lambda C^{\square\!\square\;\square\!\square\;\begin{smallmatrix}\square\\\square\end{smallmatrix}}_{\mu\;\;\nu\;\;\lambda} M_\lambda = \sum_{\lambda,i} C^{\square\!\square\;\square\!\square\;\begin{smallmatrix}\square\\\square\end{smallmatrix}}_{\mu\;\;\nu\;\;\lambda} D^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}_{i\lambda} K_i. \tag{14}$$

An analysis of this requirement leads to a chain of conclusions that we summarize briefly. The third term in (13), containing D functions of the representation $\begin{smallmatrix}\square\!\square\\\square\!\square\end{smallmatrix}$, is to vanish. Since it is not possible to choose vectors x so that this term vanishes identically as an operator, the remaining possibility

is to restrain the the space (5) of its domain to some subspace $V = \{|v\rangle\}$. More precisely, for this term to vanish, there must exist a subalgebra $\mathbf{L} \subset so(n)_K$, spanned by some $\{K_\alpha\}$, such that $K_\alpha \in \mathbf{L} \Rightarrow K_\alpha\,|v\rangle = 0$. Requiring additionally that this subspace ought to close under an action of the shear generators, and that the first two terms ought to yield (14), we arrive at the following two necessary conditions:

1. The algebra $\mathbf{L}$, must be a symmetric subalgebra of $so(n)$, i.e.

$$[\mathbf{L},\mathbf{N}] \subset \mathbf{N}, [\mathbf{N},\mathbf{N}] \subset \mathbf{L}; \mathbf{N} = \mathbf{L}^{\perp}, \tag{15}$$

2. The vector $\left|\begin{array}{c}\square\!\square\\ x\end{array}\right\rangle$ ought to be invariant under the L subgroup action (subgroup of $Spin(n)$ corresponding to $\mathbf{L}$), i.e.

$$K_\alpha \in \mathbf{L} \Rightarrow K_\alpha \left|\begin{array}{c}\square\!\square\\ x\end{array}\right\rangle = 0. \tag{16}$$

The space V is thus $Spin(n)/L$. In Wigner's terminology, this means that L is the little group of the contracted algebra representation, and that necessarily it is to be represented trivially. Besides, the little group is to be a symmetric subgroup of the $Spin(n)$ group. This coincides with one class of the solutions found by Hermann [3]. However, now we demonstrated that there are no other solutions in the $sl(n,\mathbb{R})$ algebra cases, in particular, there are no solutions with little group represented non trivially.

As for the first requirement, an inspection of the tables of symmetric spaces, yields two possibilities: $L = Spin(m)\times Spin(n-m)$, where $Spin(1) \equiv 1$, and, for $n = 2k$, $L = U(k)$ (U is the unitary group). However, this second possibility certainly does not imply another solution, since it turns out that there is no vector satisfying the second above property.

Thus, *the only remaining possibility* is as follows,

$$L = Spin(m) \times Spin(n-m), \quad m = 1, 2, \ldots, n-1 \quad Spin(1) \equiv 1. \tag{17}$$

It is not difficult to show that proportionality of (13) and (14) really holds in this case. The vector $\left|\begin{array}{c}\square\!\square\\ x\end{array}\right\rangle$ exists, and it is the one corresponding to traceless diagonal $n\times n$ matrix $diag(\frac{1}{m},\ldots,\frac{1}{m},-\frac{1}{n-m},\ldots,-\frac{1}{n-m})$.

4 Matrix elements

The approach presented in this paper allows us also to write down explicitly the matrix elements of the $sl(n,\mathbb{R})$ generators in the cases when the Gell-Mann formula is valid. The possible cases are determined by the numbers n and m. The corresponding representation space (not irreducible in general) is the one over the coset space $Spin(n)/Spin(m) \times Spin(n-m)$. The proportionality factor α is determined to be:

$$\alpha = \frac{1}{2}\sqrt{\frac{m(n-m)}{n}}, \tag{18}$$

and, in a matrix notation for $\square\square$ representation:

$$\left| \begin{array}{c} \square\square \\ x \end{array} \right\rangle = \sqrt{\tfrac{m(n-m)}{n}}\, diag(\frac{1}{m}, \ldots, \frac{1}{m}, -\frac{1}{n-m}, \ldots, -\frac{1}{n-m}). \tag{19}$$

The Gell-Mann formula (3,11), and the matrix representation of the contracted Abelian generators U (7) yield:

$$\begin{gathered} \left\langle \begin{array}{c} J' \\ m' \end{array} \right| T_\mu \left| \begin{array}{c} J \\ m \end{array} \right\rangle = \\ \sqrt{\tfrac{m(n-m)}{4n}} \sqrt{\tfrac{dim(J)}{dim(J')}} \left(C_2(J') - C_2(J) + \sigma\right) C^{\,J\,\square\square\,J'}_{\;0\;\;0\;\;0}\; C^{\,J\,\square\square\,J'}_{\;m\;\;\mu\;\;m'}\,. \end{gathered} \tag{20}$$

The zeroes in the Clebsch-Gordan coefficient here denote vectors that are invariant w.r.t. $Spin(m) \times Spin(n-m)$ transformations (in that spirit $\left| \begin{array}{c} \square\square \\ x \end{array} \right\rangle = \left| \begin{array}{c} \square\square \\ 0 \end{array} \right\rangle$). In the formula (20), the space reduction from $\mathcal{L}^2(Spin(n))$ to $\mathcal{L}^2(Spin(n)/Spin(m) \times Spin(n-m))$ implies a reduction of the basis (5), i.e. $\left| \begin{array}{c} J \\ 0\,m \end{array} \right\rangle \to \left| \begin{array}{c} J \\ m \end{array} \right\rangle$ (only the vectors invariant w.r.t. left $Spin(m) \times Spin(n-m)$ action remain).

The expression (20), together with the action of the $Spin(n)$ generators (6) provides an explicit form of the $SL(n,\mathbb{R})$ generators representation, that is labeled by a free parameter σ. Such representations are multiplicity free w.r.t. the maximal compact $Spin(n)$ subgroup, and all of them are tensorial.

5 Conclusion

In this paper, we clarified the issue of the Gell-Mann formula validity for the $sl(n, \mathbb{R}) \to r_{\frac{n(n+1)}{2}-1} \uplus so(n)$ algebra contraction. We have shown that the only $sl(n, \mathbb{R})$ representations obtainable in this way are given in Hilbert spaces over the symmetric spaces $Spin(n)/Spin(m) \times Spin(n-m)$, $m = 1, 2, \ldots, n-1$. Moreover, by making use of the Gell-Mann formula in these spaces, we have obtained a closed form expressions of the noncompact operators (generating $SL(n, \mathbb{R})/SO(n)$ cosets) irreducible representations matrix elements. The matrix elements of both compact and noncompact operators of the $sl(n, \mathbb{R})$ algebra are given by (6) and (20), respectively. In particular, it turns out that, due to Gell-Mann's formula validity conditions, no representations with $so(n)$ subalgebra representations multiplicity can be obtained in this way. Moreover, the matrix expressions of the noncompact operators as given by (20) do not account for the $sl(n, \mathbb{R})$ spinorial representations. Due to mutual connection of the $sl(n, \mathbb{R})$ and $su(n)$ algebras, the results of this paper apply to the corresponding $su(n)$ case as well. The $SU(n)/SO(n)$ generators differ from the corresponding $sl(n, \mathbb{R})$ operators by the imaginary unit multiplicative factor, while the spinorial representations issue in the $su(n)$ case is pointless due to the fact that the $SU(n)$ is a simply connected (there exists no double cover) group.

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